Counterrotating Dust Disk Around a Schwarzschild Black Hole: New Fully Integrated Explicit Exact Solution

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The first fully integrated explicit exact solution of the Einstein field equations corresponding to the superposition of a counterrotating dust disk with a central black hole is presented. The obtained solution represents an infinite annular thin disk (a disk with an inner edge) around the Schwarzschild black hole. The mass of the disk is finite and the energy-momentum tensor agrees with all the energy conditions. Furthermore, the total mass of the disk when the black hole is present is less than the total mass of the disk alone. The solution can also be interpreted as describing a thin disk made of two counterrotanting dust fluids that are also in agreement with all the energy conditions. Additionally, as we will show shortly in a subsequent paper, the above solution is the first one of an infinite family of solutions.

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The observational data supporting the existence of black holes at the nucleus of some galaxies, including the Milky Way, is today so abundant that there is no doubt about the relevance of the study of binary systems composed by a thin disk surrounding a central black hole. Accordingly, a lot of work has been developed in the last years in order to obtain a better understanding of the different aspects involved in the dynamics of these systems (see [1] for a recent review of the main works). Now, due to the presence of a black hole, the gravitational fields involved are so strong that the proper theoretical framework to analytically study this configurations is provided by the general theory of relativity. Therefore, a strong effort has been dedicated to the obtention of exact solutions of Einstein equations corresponding to thin disklike sources with a central black hole. However, until now, any explicit exact solution corresponding to such kind of systems has been obtained.

Stationary and axially symmetric solutions of the Einstein equations are the best choice to attempt to describe the gravitational fields of disks around black holes in an exact analytical manner. At the same time, such spacetimes are of obvious astrophysical importance, as they describe the exterior of equilibrium configurations of bodies in rotation. So, through the years, several examples of solutions corresponding to black holes or to thin disklike sources has been obtained by many different techniques. However, due to the nonlinear character of the Einstein equations, solutions corresponding to the superposition of black holes and thin disks are not so easy to obtain and so, until now, exact stationary solutions have not been obtained.

On the other hand, if we only consider static configurations, the line element is characterized only by two metric functions. Furthermore, in the vacuum case, the Einstein equations system implies that one of the metric functions satisfies the Laplace equation whereas that the other one can be obtained by quadratures. Therefore, as a consequence of the linearity of the Laplace equation, solutions corresponding to the superposition of thin disks and black holes can be, in principle, easily obtained. However, only very few solutions has been obtained and neither of them has been fully explicitly integrated [2, 3, 4, 5, 6, 7, 8, 9].

In this letter we present what seems to be the first fully integrated explicit exact solution for the superposition of a thin disk and a black hole. We begin by considering the Weyl metric for a static axially symmetric spacetime, written as [10]

$$ds^2 = -e^{2\Phi}dt^2 + e^{-2\Phi}[r^2 d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)], \quad (1)$$

with Φ and Λ only depending on r and z. The Einstein vacuum equations leads to the Laplace equation for Φ,

$$\Phi_{,rr} + \frac{1}{r}\Phi_{,r} + \Phi_{,zz} = 0, \quad (2)$$

and, given Φ, Λ is obtained by solving the quadrature

$$\Lambda[\Phi] = \int r[(\Phi_{,r}^2 - \Phi_{,z}^2)dr + 2\Phi_{,z}\Phi_{,z}dz], \quad (3)$$

whose integrability is granted by equation (2).

We consider a solution of (2) of the form

$$\Phi = \psi + \phi, \quad (4)$$

where ψ corresponds to a black hole solution whereas that ϕ corresponds to a thin disk solution. Now, by using (4) in (3), we obtain

$$\Lambda[\Phi] = \Lambda[\psi] + \Lambda[\phi] + 2\Lambda[\psi, \phi], \quad (5)$$

with

$$\begin{aligned}\Lambda[\psi, \phi] \;=\; & \int r[(\psi_{,r}\phi_{,r} - \psi_{,z}\phi_{,z})dr \\ & +(\psi_{,r}\phi_{,z} + \psi_{,z}\phi_{,z})dz],\end{aligned} \quad (6)$$

a term due to the nonlinear character of (3). For the black hole we take ψ and $\Lambda[\psi]$ as given by the Schwarszchild solution written as

$$\psi = \frac{1}{2}\ln\left[\frac{\zeta-1}{\zeta+1}\right], \tag{7}$$

$$\Lambda[\psi] = \frac{1}{2}\ln\left[\frac{\zeta^2-1}{\zeta^2-\eta^2}\right], \tag{8}$$

where the prolate spheroidal coordinates are defined by means of

$$r^2 = m^2(\zeta^2-1)(1-\eta^2), \qquad z = m\zeta\eta \tag{9}$$

and $1 \le \zeta < \infty$, $-1 \le \eta \le 1$.

Now, in order to obtain the thin disk solution, we introduce the oblate spheroidal coordinates by means of

$$r^2 = a^2(x^2+1)(1-y^2), \qquad z = axy, \tag{10}$$

with the ranges taken as $-\infty < x < \infty$, $0 \le y \le 1$. The disk is obtained by taking $y = 0$ and so is located at $z = 0$, $r \ge a$. On crossing the disk, x changes sign but does not change in absolute value, so that an even function of x is a continuous function everywhere but has a discontinuous x derivative at the disk. By solving the Laplace equation (2), with the proper boundary conditions corresponding to a thin disk with an inner edge, we obtain for ϕ the simple expression

$$\phi = \frac{\alpha y}{a(x^2+y^2)}. \tag{11}$$

Then, after a simple integration of (3), we obtain for $\Lambda[\phi]$ the expression

$$\Lambda[\phi] = -\frac{\alpha^2(1-y^2)A(x,y)}{4a^2(x^2+y^2)^4}, \tag{12}$$

where

$$A(x,y) = x^4(9y^2-1) + 2x^2y^2(y^2+3) + y^4(y^2-1),$$

with α an arbitrary constant and a the inner radius of the disk.

The surface energy-momentum tensor of the disk can be written as [11]

$$S_{ab} = \epsilon V_a V_b, \tag{13}$$

where $V^a = e^{-\Phi}\delta^a_{\ 0}$. The surface energy density is given by

$$\epsilon = \frac{4\alpha}{a^2x^3}\exp\left\{-\frac{\alpha^2}{4a^2x^4}\right\}, \tag{14}$$

for $x \ge 0$, and will be allways positive if we take $\alpha > 0$. We then have a dust disk in agreement with all the energy conditions. The total mass of the disk can be easily computed and we obtain

$$\frac{M}{2\pi} = \int_a^\infty \epsilon(r) r dr = \sqrt{2a\alpha}\,\Gamma(1/4), \tag{15}$$

so that the disk is of infinite extension but with finite mass. In Fig. 1 we plot the energy density for some values of α.

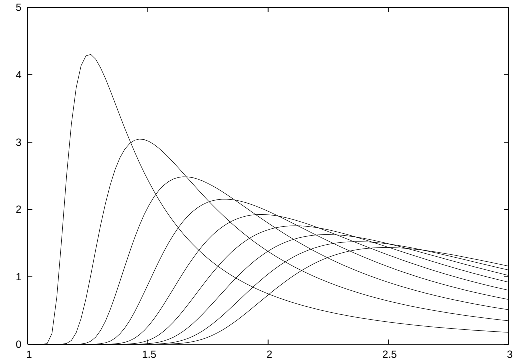

FIG. 1: Energy density $\tilde{\epsilon} = a\epsilon$ as a function of $\tilde{r} = r/a$ for $\tilde{\alpha} = \alpha/a = 1, ..., 9$. The first curve on left corresponds to $\tilde{\alpha} = 1$, whereas that the last curve on right corresponds to $\tilde{\alpha} = 9$.

For the combined black hole and disk system, the integration of the mixed term $\Lambda[\phi,\psi]$ in (5) can be done with aid of some very useful computational techniques introduced in [12]. So, we obtain

$$\Lambda[\phi,\psi] = \frac{\alpha}{2}\left[\frac{1-y}{x^2+y^2}\right](\Lambda_1 - \Lambda_2), \tag{16}$$

where

$$\Lambda_1 = \frac{ay(1-y)(1+x^2) - mx(1+y)(\zeta-1)(1-\eta)}{[ax+m(1-\zeta-\eta)]^2 + a^2(1-y)^2},$$

$$\Lambda_2 = \frac{ay(1-y)(1+x^2) - mx(1+y)(\zeta+1)(1-\eta)}{[ax-m(1+\zeta-\eta)]^2 + a^2(1-y)^2},$$

that vanishes at the axis, when $y = 1$, and at the disk, when $y = \eta = 0$. More details about the obtention of the solution will be presented in a subsequent paper.

The presence of the black hole modifies the energy-momentum tensor of the disk, in such a way that arises a nonzero pressure at the azimuthal direction. Accordingly, the modified energy-momentum tensor of the disk can be written as [11]

$$\mathcal{S}_{ab} = \varepsilon V_a V_b + p X_a X_b, \tag{17}$$

where $X^a = e^{\Phi}\delta^a_{\ 1}$. The energy density, the azimuthal pressure and the "effective Newtonian density" are given,

respectively, by

$$\varepsilon = \left[\frac{\zeta - 1}{\zeta + 1}\right]\epsilon, \qquad (18)$$

$$p = \left[\frac{1}{\zeta + 1}\right]\epsilon, \qquad (19)$$

$$\sigma = \left[\frac{\zeta}{\zeta + 1}\right]\epsilon. \qquad (20)$$

where $\sigma = \varepsilon + p$. Now, as $\epsilon \geq 0$ and $\zeta \geq 1$, ε and σ will be positives everywhere. Accordingly, the energy-momentum tensor will be in fully agreement with the weak and strong energy conditions. On the other hand, in order to fulfill the dominant energy condition, we require that $p \leq \varepsilon$, which implies that $\zeta \geq 2$ and $a \geq \sqrt{3}m$. Furthermore, we have that $\sigma < \epsilon$, and thus the total mass of the disk when the black hole is present is less than the total mass of the disk alone.

The energy-momentum tensor can also be interpreted as the superposition of two counterrotating fluids. In order to do this, we cast $\mathcal{S}^{ab}$ as [11]

$$\mathcal{S}^{ab} = \varepsilon_+ U^a_+ U^b_+ + \varepsilon_- U^a_- U^b_-, \qquad (21)$$

where

$$\varepsilon_+ = \varepsilon_- = \left[\frac{\zeta - 2}{\zeta + 1}\right]\frac{\epsilon}{2}, \qquad (22)$$

are the energy densities of the two counterrotating fluids. The counterrotating velocity vectors are given by [11]

$$U^a_\pm = \frac{V^a \pm U X^a}{\sqrt{1 - U^2}}, \qquad (23)$$

where

$$U^2 = \frac{p}{\varepsilon} = \frac{1}{\zeta - 1} \leq 1, \qquad (24)$$

is the counterrotating tangential velocity. So, we have two counterrotating dust fluids with equal energy densities. Now, as $\varepsilon_\pm \geq 0$, the two counterrotating dust disks are in fully agreement with all the energy conditions.

As we can see, the above solution presents some very interesting properties. First, the associated material source presents a very reasonable behavior and its energy momentum tensor is in agreement with all the energy conditions. Furthermore, its relative simplicity when expressed in terms of the spheroidal coordinates, prolates and oblates, makes it very easy to study different dynamical aspects, like the motion of particles inside and outside the disk, the stability of the orbits and the possible existence of singularities. Additionally, as we will show shortly in a subsequent paper, the above solution is the first one of an infinite family of solutions, all of them having many remarkable properties in common with the solution here presented.

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